arXiv:hep-th/0011049v1 8 Nov 2000

The Volume Element of Space-Time and Scale Invariance

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February 1, 2008

Abstract

Scale invariance is considered in the context of gravitational theories where the action, in the first order formalism, is of the form $S = \int L_1 \Phi d^4x + \int L_2 \sqrt{-g} d^4x$ where the volume element Φd^4x is independent of the metric. For global scale invariance, a "dilaton" ϕ has to be introduced, with non-trivial potentials $V(\phi) = f_1 e^{\alpha\phi}$ in L_1 and $U(\phi) = f_2 e^{2\alpha\phi}$ in L_2. This leads to non-trivial mass generation and a potential for ϕ which is interesting for inflation. Interpolating models for natural transition from inflation to a slowly accelerated universe at late times appear naturally. This is also achieved for "Quintessential models", which are scale invariant but formulated with the use of volume element Φd^4x alone. For closed strings and branes (including the supersymmetric cases), the modified measure formulation is possible and does not require the introduction of a particular scale (the string or brane tension) from the begining but rather these appear as integration constants.

1 The Simplest Scalar-Gravity Model, in the absence of fermions

The concept of scale invariance appears as an attractive possibility for a fundamental symmetry of nature. In its most naive realizations, such a

symmetry is not a viable symmetry, however, since nature seems to have chosen some typical scales.

Here we will find that scale invariance can nevertheless be incorporated into realistic, generally covariant field theories. However, scale invariance has to be discussed in a more general framework than that of standard generally relativistic theories, where we must allow in the action, in addition to the ordinary measure of integration $\sqrt{-g}d^4x$, another one, Φd^4x, where Φ is a density built out of degrees of freedom independent of the metric.

For example, given 4-scalars φ_a (a = 1,2,3,4), one can construct the density

$$\Phi = \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}\partial_\mu\varphi_a\partial_\nu\varphi_b\partial_\alpha\varphi_c\partial_\beta\varphi_d \tag{1}$$

One can allow both geometrical objects in the theory and consider [1] ,

$$S = \int L_1\Phi d^4x + \int L_2\sqrt{-g}d^4x \tag{2}$$

Here L_1 and L_2 are φ_a independent. There is a good reason not to consider mixing of Φ and $\sqrt{-g}$, like for example using $\frac{\Phi^2}{\sqrt{-g}}$. This is because (2) is invariant (up to the inte divergence) under the infinite dimensional symmetry $\varphi_a \rightarrow \varphi_a + f_a(L_1)$ where $f_a(L_1)$ is an arbitrary function of L_1 if L_1 and L_2 are φ_a independent. Such symmetry (up to the integral of a total divergence) is absent if mixed terms are present.

We will study now the dynamics of a scalar field ϕ interacting with gravity as given by the action (2) with [2,3,4]

$$L_1 = \frac{-1}{\kappa}R(\Gamma, g) + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi), L_2 = U(\phi) \tag{3}$$

$$R(\Gamma, g) = g^{\mu\nu}R_{\mu\nu}(\Gamma), R_{\mu\nu}(\Gamma) = R^\lambda_{\mu\nu\lambda}, R^\lambda_{\mu\nu\sigma}(\Gamma) = \Gamma^\lambda_{\mu\nu,\sigma} - \Gamma^\lambda_{\mu\sigma,\nu} + \Gamma^\lambda_{\alpha\sigma}\Gamma^\alpha_{\mu\nu} - \Gamma^\lambda_{\alpha\nu}\Gamma^\alpha_{\mu\sigma}. \tag{4}$$

In the variational principle $\Gamma^\lambda_{\mu\nu}, g_{\mu\nu}$, the measure fields scalars φ_a and the scalar field ϕ are all to be treated as independent variables. If we perform the global scale transformation (θ = constant)

$$g_{\mu\nu} \rightarrow e^\theta g_{\mu\nu} \tag{5}$$

then (2), with the definitions (3), (4), is invariant provided $V(\phi)$ and $U(\phi)$ are of the form

$$V(\phi) = f_1 e^{\alpha\phi}, U(\phi) = f_2 e^{2\alpha\phi} \tag{6}$$

and φ_a is transformed according to $\varphi_a \to \lambda_a \varphi_a$ (no sum on a) which means $\Phi \to \left(\prod_a \lambda_a\right)\Phi \equiv \lambda\Phi$ such that $\lambda = e^\theta$ and $\phi \to \phi - \frac{\theta}{\alpha}$. In this case we call the scalar field ϕ needed to implement scale invariance "dilaton".

1.1 Equations of Motion

Let us consider the equations which are obtained from the variation of the φ_a fields. We obtain then $A^\mu_a \partial_\mu L_1 = 0$ where $A^\mu_a = \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}\partial_\nu\varphi_b\partial_\alpha\varphi_c\partial_\beta\varphi_d$. Since det $(A^\mu_a) = \frac{4^{-4}}{4!}\Phi^3 \neq 0$ if $\Phi \neq 0$. Therefore if $\Phi \neq 0$ we obtain that $\partial_\mu L_1 = 0$, or that $L_1 = M$, where M is constant. This constant M appears in a self-consistency condition of the equations of motion that allows us to solve for $\chi \equiv \frac{\Phi}{\sqrt{-g}}$

$$\chi = \frac{2U(\phi)}{M + V(\phi)}. \tag{7}$$

To get the physical content of the theory, it is convenient to go to the Einstein conformal frame where

$$\overline{g}_{\mu\nu} = \chi g_{\mu\nu} \tag{8}$$

and χ given by (7). In terms of $\overline{g}_{\mu\nu}$ the non Riemannian contribution (defined as $\Sigma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \{^{\lambda}_{\mu\nu}\}$ where $\{^{\lambda}_{\mu\nu}\}$ is the Christoffel symbol), disappears from the equations, which can be written then in the Einstein form ($R_{\mu\nu}(\overline{g}_{\alpha\beta}) =$ usual Ricci tensor)

$$R_{\mu\nu}(\overline{g}_{\alpha\beta}) - \frac{1}{2}\overline{g}_{\mu\nu}R(\overline{g}_{\alpha\beta}) = \frac{\kappa}{2}T^{eff}_{\mu\nu}(\phi) \tag{9}$$

where

$$T^{eff}_{\mu\nu}(\phi) = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\phi_{,\alpha}\phi_{,\beta}\overline{g}^{\alpha\beta} + \overline{g}_{\mu\nu}V_{eff}(\phi), V_{eff}(\phi) = \frac{1}{4U(\phi)}(V+M)^2. \tag{10}$$

If $V(\phi) = f_1 e^{\alpha\phi}$ and $U(\phi) = f_2 e^{2\alpha\phi}$ as required by scale invariance, we obtain from (10)

$$V_{eff} = \frac{1}{4f_2}(f_1 + Me^{-\alpha\phi})^2 \tag{11}$$

Since we can always perform the transformation $\phi \rightarrow -\phi$ we can choose by convention $\alpha > O$. We then see that as $\phi \rightarrow \infty, V_{eff} \rightarrow \frac{f_1^2}{4f_2}$ = const. providing an infinite flat region. Also a minimum is achieved at zero cosmological constant , without fine tuning for the case $\frac{f_1}{M} < O$ at the point $\phi_{min} = \frac{-1}{\alpha} ln \mid \frac{f_1}{M} \mid$. Finally, the second derivative of the potential V_{eff} at the minimum is $V''_{eff} = \frac{\alpha^2}{2f_2} \mid f_1 \mid^2 > O$

2 Some Physics of the Model: Inflation, Connection to Zee's Induced Gravity Model and Possible applications to the Present state of the Universe

A very important point to be raised is that since there is an infinite region of flat potential for $\phi \rightarrow \infty$, we expect a slow rolling inflationary scenario to be viable, provided the universe is started at a sufficiently large value of the scalar field ϕ for example.

The fact that there is a flat region is directly correlated to the fact that there is scale invariance. In fact, in terms of $\overline{g}_{\mu\nu}$ and ϕ, the scale transformations affect only ϕ ($\overline{g}_{\mu\nu}$ is scale invariant) and it is simply a translation in the scalar field space. The flat region reflects a translation invariant region, where therefore scale invariance is restored. By contrast any non trivial shape of the potential means ssb of scale invariance, as is the case in a region of the potential (for $M \neq 0$).

It is also very interesting to notice that the theory can be related to the induced gravity theory of Zee [5], defined by the action,

$$S = \int \sqrt{-g}(-\frac{1}{2}\epsilon\varphi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{\lambda}{8}(\varphi^2 - \eta^2)^2)d^4x \tag{12}$$

Here it is assumed that the second order formalism is used, i.e. $R = R(g)$ = usual Riemannian scalar curvature defined in terms of $g_{\mu\nu}$. Notice that if $\eta = 0$, the action is invariant under the global scale transformation $g_{\mu\nu} \rightarrow e^\theta g_{\mu\nu}$, $\varphi \rightarrow e^{-\frac{\theta}{2}}\varphi$, but a finite induced Newton's constant is defined only if η is non vanishing. Then defining ($2k^2 = \kappa$) $\overline{g}_{\mu\nu} = k^2\epsilon\varphi^2 g_{\mu\nu}$ and the scalar field $\phi = \frac{1}{k}\sqrt{6 + \frac{1}{\epsilon}} ln\varphi$, one can then show that the induced gravity

model is equivalent to standard General Relativity (expressed in terms of $\overline{g}_{\mu\nu}$) minimally coupled to the scalar field ϕ which has a potential [6] , $V_{eff} = \frac{\lambda}{8k^4\epsilon^2}(1-\eta^2 e^{-2\sqrt{\frac{\epsilon}{1+6\epsilon}}k\phi})^2$ which is exactly the form (11) with $\alpha = 2\sqrt{\frac{\epsilon}{1+6\epsilon}}k$ (in Ref.6, k is called κ). The induced gravity model (12) is quite successful from the point of view of its applications to inflation and it has been studied by a number of authors in this context [7] . Notice that the induced gravity model is not consistent with scale invariance for a non vanishing η, while the theory developed here, which leads to the induced gravity model after ssb, has been constructed starting with scale invariance as a fundamental principle.

A similar thing happens when we take the pure gravity form (see Refs. 8, 9 and 10), $S = \frac{1}{2}\int\sqrt{-g}(R+\beta R^2)d^4x$ Here again it is assumed that the second order formalism is used, i.e. $R = R(g)$ = usual Riemannian scalar curvature defined in terms of $g_{\mu\nu}$. Notice that if only the βR^2 term is present, the action is invariant under the global scale transformation $g_{\mu\nu} \to e^{\theta} g_{\mu\nu}$.

Then defining $\overline{g}_{\mu\nu} = [1+2\beta R(g)]g_{\mu\nu}$ and $\phi = \sqrt{\frac{3}{2}}ln[1+2\beta R(g)]$, one can then show that the βR^2 model is equivalent to standard General Relativity (expressed in terms of $\overline{g}_{\mu\nu}$) coupled to a minimally coupled scalar field which has a potential (see for example Ref. 9) $V_{eff} = \frac{1}{8\beta}(1-e^{-\sqrt{\frac{2}{3}}\phi})^2$, which is exactly the form (11) for a very special choice of α (= $\sqrt{\frac{2}{3}}$ in Planck units).

Notice that as R^2 dominates, $\phi \to \phi + const.$ under a dilatation transformation and one can again understand the flat region as a consequence of scale invariance in some limit.

Density fluctuations have been studied in the βR^2 model [10] and it was found that $10^{11}Gev < \sqrt{\frac{1}{\beta}} < 10^{13}Gev$ gives acceptable density fluctuations. Notice that when changing continously the parameters in the Zee model, we obtain a correspondence with with the theory defined here for a continous range of the α parameter, while for the βR^2 the correspondence is achieved for a very specific value of α only.

Furthermore, independently of the question of what kind of models can be equivalent (before we couple it to matter) to the scale invariant theory defined here, one can consider this model as suitable for the present day universe rather than for the early universe, after we suitably reinterpret the meaning of the scalar field ϕ. This can provide a long lived almost constant vacuum energy for a long period of time, which can be small if $f_1^2/4f_2$ is small. Such small energy density will eventually disappear when the universe

achieves its true vacuum state.

Notice that a small value of $\frac{f_1^2}{f_2}$ can be achieved if we let $f_2 >> f_1$. In this case $\frac{f_1^2}{f_2} << f_1$, i.e. a very small scale for the energy density of the universe is obtained by the existence of a very high scale (that of f_2) the same way as a small fermion mass is obtained in the see-saw mechanism [11] from the existence also of a large mass scale.

3 The Introduction of Fermions

Since in nature there is more than just scalars and gravity, it is necessary to consider the extension of the model so as to accomodate fermions. Taking, for example, the case of a fermion ψ, where the kinetic term of the fermion is chosen to be part of L_1

$$S_{fk} = \int L_{fk} \Phi d^4x \tag{13}$$

$$L_{fk} = \frac{i}{2}\overline{\psi}[\gamma^a V_a^\mu(\overrightarrow{\partial}_\mu + \frac{1}{2}\omega_\mu^{cd}\sigma_{cd}) - (\overleftarrow{\partial}_\mu + \frac{1}{2}\omega_\mu^{cd}\sigma_{cd})\gamma^a V_a^\mu]\psi \tag{14}$$

there V_a^μ is the vierbein, $\sigma_{cd} = \frac{1}{2}[\gamma_c, \gamma_d]$, the spin connection ω_μ^{cd} is determined by variation with respect to ω_μ^{cd} and, for self-consistency, the curvature scalar is taken to be (if we want to deal with ω_μ^{ab} instead of $\Gamma_{\mu\nu}^\lambda$ everywhere)

$$R = V^{a\mu}V^{b\nu}R_{\mu\nu ab}(\omega), R_{\mu\nu ab}(\omega) = \partial_\mu\omega_{\nu ab} - \partial_\nu\omega_{\mu ab} + (\omega_{\mu a}^c\omega_{\nu cb} - \omega_{\nu a}^c\omega_{\mu cb}). \tag{15}$$

Global scale invariance is obtained provided ψ also transforms, as in $\psi \to \lambda^{-\frac{1}{4}}\psi$. Mass term consistent with scale invariance exist,

$$S_{fm} = m_1 \int \overline{\psi}\psi e^{\alpha\phi/2}\Phi d^4x + m_2 \int \overline{\psi}\psi e^{3\alpha\phi/2}\sqrt{-g}d^4x. \tag{16}$$

If we consider the situation where $m_1 e^{\alpha\phi/2}\overline{\psi}\psi$ or $m_2 e^{3\alpha\phi/2}\overline{\psi}\psi$ are much bigger than $V(\phi)$ + M, i.e. a high density approximation, we obtain that instead of (7) that the consistency condition is [3] $(3m_2 e^{3\alpha\phi/2} + m_1 e^{\alpha\phi/2}\chi)\overline{\psi}\psi = 0$, which means $\chi = -\frac{3m_2}{m_1}e^{\alpha\phi}$. Using this in (16), we obtain, after going to the conformal Einstein frame, which involves, also a transformation of the fermion fields, necessary so as to achieve Einstein-Cartan form for both the

gravitational and fermion equations. These transformations are, $\overline{g}_{\mu\nu} = \chi g_{\mu\nu}$ (or $\overline{V}^a_\mu = \chi^{\frac{1}{2}} V^a_\mu$) and $\psi' = \chi^{-\frac{1}{4}}\psi$ and they lead to a mass term,

$$S_{fm} = -2m_2(\frac{|m_1|}{3|m_2|})^{3/2} \int \sqrt{-\overline{g}}\overline{\psi}'\psi' d^4x \tag{17}$$

The ϕ dependence of the mass term has disappeared, i.e. masses are constants.

There is one situation where the low density of matter can also give results which are similar to those obtained in the high density approximation, in that the coupling of the ϕ field disappears and that the mass term becomes of a conventional form in the Einstein conformal frame.

This is the case, when we study the theory for the limit $\phi \to \infty$. Then $U(\phi) \to \infty$ and $V(\phi) \to \infty$. In this case, taking $m_1 e^{\alpha\phi/2}\overline{\psi}\psi$ and $m_2 e^{3\alpha\phi/2}\overline{\psi}\psi$ much smaller than $V(\phi)$ or $U(\phi)$ respectively, therefore one can see that (7) is a good approximation and since also M can be ignored in the self consistency condition (7) in this limit, we get then, $\chi = \frac{2f_2}{f_1}e^{\alpha\phi}$. If this is inserted in (16), we get $S_{fm} = m\int\sqrt{-\overline{g}}\overline{\psi}'\psi' d^4x$, where

$$m = m_1(\frac{f_1}{2f_2})^{\frac{1}{2}} + m_2(\frac{f_1}{2f_2})^{\frac{3}{2}} \tag{18}$$

Comparing (17) and (18) and taking m_1 and m_2 of the same order of magnitude, we see that the mass of the Dirac particle is much smaller in the region $\phi \to \infty$, for which (18) is valid, than it is in the region of high density of the Dirac particle relative to $V(\phi) + M$, as displayed in eq. (17), if the "see-saw" assumption $\frac{f_1}{f_2} << 1$ is made. Therefore if space is populated by these diluted Dirac particles of this type, the mass of these particles will grow substantially if we go to the true vacuum state, valid in the absence of matter, i.e. $V + M = 0$, as dictated by V_{eff} given by eq. (11).

The presence of matter pushes therefore the minimum of energy to a state where $V + M > 0$. The real vacuum in the presence of matter should not be located in the region $\phi \to \infty$, which minimizes the matter energy, but maximizes the potential energy V_{eff} and not at $V + M = 0$, which minimizes V_{eff}, and where particle masses are big, but somewhere in a balanced intermediate stage. Clearly how much above $V + M = 0$ such true vacuum is located must be correlated to how much particle density is there in the Universe. A non zero vacuum energy, which must be of the same order of the particle energy

density, has to appear and this could explain the "accelerated universe" that appears to be implied by the most recent observations, together with the "cosmic coincidence", that requires the vacuum energy be of the same order of magnitude to the matter energy [12] .

4 On The Absence of the Goldstone Boson

It is worthwhile to point out that in the models with scale invariance dicussed here there is no Goldstone boson, when we look at the excitations arround the true vacuum with zero cosmological constant. The basic reason that Goldstone's theorem does not apply is that although there is a global symmetry, which leads, according to Noether's theorem to a locally conserved current, the spatial components of such current have an infrared singular behavior, leading to flux leaking through infinity and to a non conservation of the would be dilaton charge [4] .

Let us see that this is indeed the case and for this purpose, let us ignore the fermions. Since there is a symmetry according to Noether's theorem, there is a conserved current given by (since the variation of the lagrangian density vanishes under the scale symmetry),

$$j^\mu = \frac{\partial L}{\partial(\partial_\mu \varphi_a)}\delta\varphi_a + \frac{\partial L}{\partial(\partial_\mu \phi)}\delta\phi \tag{19}$$

since in the first order formalism $\frac{\partial L}{\partial(\partial_\mu g_{\alpha\beta})} = 0$ and $\delta\Gamma^\lambda_{\mu\nu} = 0$ under the scale symmetry defined before.

Let us now consider what we should take for $\delta\varphi_a$. As part of the dilatation symmetry, we have that $\varphi_a \to \lambda_a \varphi_a$ (no sum on a) and since $\left(\prod_a \lambda_a\right) \equiv \lambda = e^\theta$, we have, taking a transformation infinitesimally close to the identity, i.e. $\lambda_a = 1 + \epsilon_a$, with $\epsilon_a << 1$ and all ϵ_a equal, so that $\epsilon_a = \theta/4$ and since also $\delta\phi = -\frac{\theta}{\alpha}$, that the conserved dilatation current is,

$$j^\mu_\theta = -\frac{\theta}{\alpha}\Phi\partial^\mu\phi + \theta\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}\varphi_a\partial_\nu\varphi_b\partial_\alpha\varphi_c\partial_\beta\varphi_d L_1 \equiv \theta j^\mu_D \tag{20}$$

To see the basic reasons why the dilatation current has an infrared singular behavior, let us consider the spatial behavior of the φ_a fields for the case

of a simple spatially flat Robertson-Walker solution of the form

$$ds^2 = -dt^2 + R^2(t)(dx^2 + dy^2 + dz^2), \phi = \phi(t) \tag{21}$$

We see also from the constraint (7) that $\chi = \chi(t)$. Then, since $\chi = \chi(t) = \frac{\Phi}{R^3(t)}$, we get that,

$$\Phi = R^3(t)\chi(t) = \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}\partial_\mu\varphi_a\partial_\nu\varphi_b\partial_\alpha\varphi_c\partial_\beta\varphi_d \tag{22}$$

This can be solved by taking

$$\varphi_1 = x, \varphi_2 = y, \varphi_3 = z, \varphi_4 = -\frac{1}{4!}\int \chi(t^{'})R^3(t^{'})dt^{'} \tag{23}$$

For this case, with a time dependent scalar field $\phi(t)$ and with φ_a given above, the spatial components of the current j^μ_D, diverge linearly as $x^i \to \infty$ ($x^1 = x, x^2 = y, x^3 = z$). In fact $j^i_D \to M x^i \chi(t) R^3(t)$ as $x^i \to \infty$ Such current does indeed give flux at infinity. The current grows linearly with distance, so that the total flux is proportional to the volume enclosed and obviously the total dilatation charge is not conserved here.

5 Interpolating Models

This kind of theories can naturally provide a dynamics that interpolates between a high energy density (associated with inflation) and a very low energy density (associated with the present universe). For this consider two scalar fields ϕ_1 and ϕ_2, with normal kinetic terms coupled to Φ as it has been done with the simpler model of just one scalar field. Introducing for ϕ_1 a potential $V_1(\phi_1) = a_1 e^{\alpha_1\phi_1}$ that couples to Φ and another $U_1(\phi_1) = b_1 e^{2\alpha_1\phi_1}$ that couples to $\sqrt{-g}$ as required by scale invariance and the potential for ϕ_2, $V_2(\phi_2) = a_2 e^{\alpha_2\phi_2}$ that couples to Φ and $U_2(\phi_2) = b_2 e^{2\alpha_2\phi_2}$ that couples to $\sqrt{-g}$, we arrive (after going through the same steps as those explained in the model with just one scalar, i.e. solving the constraint and going to the Einstein frame) at the effective potential (see the last reference of Ref.3)

$$V_{eff} = \frac{(V_1(\phi_1) + V_2(\phi_2) + M)^2}{4(U_1(\phi_1) + U_2(\phi_2))} \tag{24}$$

which introduces interactions between ϕ_1 and ϕ_2, although no interactions appeared in the original action (i.e. no direct couplings appeared). If we take then $\alpha_1\phi_1$ very big while ϕ_2 is fixed, then V_{eff} approaches the constant value $\frac{a_1^2}{4b_1}$ while if we take $\alpha_2\phi_2$ to be very big while ϕ_1 is kept fixed, then V_{eff} approaches the constant value $\frac{a_2^2}{4b_2}$. One of these flat regions of the potential can be associated with a very high energy density, associated with inflation and the other can be very small and associated with the energy density of the present universe. The effective potential (24) provides therefore a dynamics that interpolates naturally between the inflationary phase and the present slowly accelerated universe.

6 Scale Invariant Quintessential Models

One may wonder whether a model that uses only one measure, the measure Φ is possible. If we follow the most straightforward approach and take the limit in (3) $L_2 = U(\phi) \to 0$, or for the scale invariant case $f_2 \to 0$, we see that the potential in (11) forces, in this singular limit, the function $f_1 + Me^{\alpha\phi}$ to vanish, therefore killing the scalar field dynamics.

It is however possible to restore non trivial scalar field dynamics, while keeping the simple structure

$$S = \int \Phi L d^4x \tag{25}$$

which has the invariance $L \to L + constant$ and therefore shows the "principle of non gravitating vacuum energy" (i.e. the irrelevance of the origin of L) in its most pure form.

The clue to obtain non trivial scalar field dynamics (and also gauge dynamics) consists in introducing a four index field strength $F_{\mu\nu\alpha\beta} = \partial_{[\mu}A_{\nu\alpha\beta]}$, as is discussed in Ref.1.

Here we will study such kind of models, subjected to the additional requirement of scale invariance, which is the unifiying feature of all the models studied in this paper.

As we will see, such a construction naturally leads to quintessential type potentials [13] which are of interest in cosmology.

Let us define the scalar field

$$y = \frac{1}{m^2} \frac{\varepsilon^{\mu\nu\alpha\beta}}{\sqrt{-g}} \partial_\mu A_{\nu\alpha\beta} \tag{26}$$

where m is a parameter with the dimensions of a mass. Then let us take an action with the four field strength and a scalar field ϕ according to

$$L = \frac{-1}{\kappa} R(\Gamma, g) - \frac{1}{pm^{4(p-1)}} y^p + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \tag{27}$$

where the curvature scalar $R(\Gamma, g)$ is once again defined by (4), p is a dimensionless number and scale invariance requires an exponential form for the scalar field potential,

$$V(\phi) = f e^{\alpha\phi} \tag{28}$$

Under these circumtances, S as given by (25), (26), (27) and (28) will be invariant under the scale transformations

$$g_{\mu\nu} \rightarrow e^{\theta} g_{\mu\nu}, \phi \rightarrow \phi - \frac{\theta}{\alpha} \tag{29}$$

$$\varphi_a \rightarrow \lambda_a \varphi_a \tag{30}$$

(no sum on a), which means

$$\Phi \rightarrow \left(\prod_a \lambda_a \right) \Phi \equiv \lambda \Phi, where \lambda = e^{\theta} \tag{31}$$

and finally

$$A_{\nu\alpha\beta} \rightarrow e^{\theta(2-\frac{1}{p})} A_{\nu\alpha\beta} \tag{32}$$

as before the variation with respect to the fields φ_a gives rise to the equation $A^\mu_a \partial_\mu L = 0$, where $A^\mu_a = \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{abcd} \partial_\nu \varphi_b \partial_\alpha \varphi_c \partial_\beta \varphi_d$. Since det $(A^\mu_a) = \frac{4^{-4}}{4!} \Phi^3 \neq 0$ if $\Phi \neq 0$. Therefore if $\Phi \neq 0$ we obtain that $\partial_\mu L = 0$, or that

$$L = M \tag{33}$$

where M is a constant. It is clear that if $M \neq 0$, the above equation spontaneously breaks scale invariance, since L transforms under (29)-(32), while M is choosen by the boundary conditions.

The variation with respect to $g^{\mu\nu}$ gives

$$\frac{1}{\kappa}R_{\mu\nu} = -\frac{y^p}{2m^{4(p-1)}}g_{\mu\nu} + \frac{1}{2}\phi_{,\mu}\phi_{,\nu} \tag{34}$$

Since we can solve for $R = g^{\mu\nu}R_{\mu\nu}$ from (34) and insert in (33), we obtain then an equartion which does not contain the scalar curvature, which is,

$$\frac{2p-1}{pm^{4(p-1)}}y^p = V + M \tag{35}$$

The variation of the action with respect to $A_{\mu\nu\alpha}$ gives (recall that $\chi \equiv \frac{\Phi}{\sqrt{-g}}$)

$$\partial_\mu(\chi y^{p-1}\varepsilon^{\mu\nu\alpha\beta}) = 0 \tag{36}$$

which means that

$$\chi y^{p-1} = \omega m^{4(p-1)} \tag{37}$$

where ω is a dimensionless constant. If $p \neq \frac{1}{2}$ and $\omega \neq 0$ we see from eqs. (29), (30), (31) and (32) that (37) spontaneously breaks scale invariance.

To get the physical content of the theory, it is convenient to go to the Einstein conformal frame where

$$\overline{g}_{\mu\nu} = \chi g_{\mu\nu} \tag{38}$$

In terms of $\overline{g}_{\mu\nu}$ the equations can be written then in the Einstein form ($R_{\mu\nu}(\overline{g}_{\alpha\beta})$ = usual Ricci tensor)

$$R_{\mu\nu}(\overline{g}_{\alpha\beta}) - \frac{1}{2}\overline{g}_{\mu\nu}R(\overline{g}_{\alpha\beta}) = \frac{\kappa}{2}T^{eff}_{\mu\nu}(\phi) \tag{39}$$

where

$$T^{eff}_{\mu\nu}(\phi) = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\phi_{,\alpha}\phi_{,\beta}\overline{g}^{\alpha\beta} + \overline{g}_{\mu\nu}V^{(p)}_{eff}(\phi) \tag{40}$$

and the potential $V^{(p)}_{eff}(\phi)$ is given by

$$V^{(p)}_{eff} = \frac{1}{\omega m^{4(1-\frac{1}{p})}}(\frac{p}{2p-1})^{2-\frac{1}{p}}(V(\phi)+M)^{2-\frac{1}{p}} \tag{41}$$

In terms of the metric $\overline{g}^{\alpha\beta}$, and $V^{(p)}_{eff}$ defined above, the equation of motion of the scalar field ϕ takes the standard General - Relativity form

$$\frac{1}{\sqrt{-\overline{g}}}\partial_\mu(\overline{g}^{\mu\nu}\sqrt{-\overline{g}}\partial_\nu\phi) + V^{(p)\prime}_{eff}(\phi) = O. \tag{42}$$

and for the case of interest, $V(\phi) = fe^{\alpha\phi}$ so that

$$V^{(p)}_{eff} = \frac{1}{\omega m^{4(1-\frac{1}{p})}}(\frac{p}{2p-1})^{2-\frac{1}{p}}(fe^{\alpha\phi} + M)^{2-\frac{1}{p}} \tag{43}$$

Notice that $\alpha\phi \to -\infty$,

$$V^{(p)}_{eff} \to \frac{1}{\omega m^{4(1-\frac{1}{p})}}(\frac{p}{2p-1})^{2-\frac{1}{p}}(M)^{2-\frac{1}{p}} = constant. \tag{44}$$

As in our previous example, there is an asymptotically flat region, which can be the region where inflation in the early universe took place.

Notice that under a scale transformation, both M and ω transform according to

$$M \to e^{-\theta}M, \omega \to e^{(\frac{1}{p}-2)\theta}\omega \tag{45}$$

Note, for example, that the asymptotically value (44) is invariant under (45). (45) transforms one vacuum into another one.

Let us consider now the case $0 < p < \frac{1}{2}$ and $M < 0, f < 0$. Then,

$$V^{(p)}_{eff} = \frac{1}{\omega m^{4(1-\frac{1}{p})}}(\frac{p}{|2p-1|})^{2-\frac{1}{p}}(|f|e^{\alpha\phi} + |M|)^{2-\frac{1}{p}} \tag{46}$$

and as $\alpha\phi \to \infty$

$$V^{(p)}_{eff} \to Ce^{-(\frac{1}{p}-2)\alpha\phi} \tag{47}$$

(C is a constant) and since $0 < p < \frac{1}{2}$, $\frac{1}{p} - 2 > 0$ so that $V^{(p)}_{eff} \to 0$ in this limit. That is, as the magnitude of the original potential $V = |f|e^{\alpha\phi}$ goes to infinity, the effective potential approaches zero without fine tunning. Since there is also a region of constant, positive value of the potential, if $\omega > 0$, the potential in question can connect, as was the case of the interpolating model of section 5, an inflationary phase with a small (and slowly decaying in this case) cosmological term, responsible fo the slowly accelerated universe today.

Here such an effect has been obtained from spontaneous symmetry breaking of scale invariance. Kaganovich [14] has studied the possibility of obtaining quintessence models from small pertubations of the two measure model defined in section 1 where the small perturbations explicitly break scale invariance.

More details concerning the scale invariant (i.e. where all scale symmetry breaking is introduced spontaneously rather than explicitly) approach to quintessence will be given elsewhere [15].

For the case $\alpha = 0$ and $p = \frac{1}{2}$ the theory has an infinite dimensional extra symmetry. To see this notice first that if $\alpha = 0$ the term $\int V \Phi d^4x = V \int \Phi d^4x$, since V is in such a case a constant. So this term is dynamically irrelevant, it is the integral of a total divergence.

The rest of the action is invariant under the infinite dimensional group of diffeomorphisms in the internal space of the φ_a

$$\varphi_a \rightarrow \varphi'_a(\varphi_b) \tag{48}$$

so that

$$\Phi \rightarrow J\Phi \tag{49}$$

where J is the jacobian of the transformation (48).

The internal diffeomorphism (48) has to be performed together with the conformal transformation of the metric, while the three index potential remains unchanged.

$$g_{\mu\nu} \rightarrow J g_{\mu\nu}, A_{\mu\nu\alpha} \rightarrow A_{\mu\nu\alpha} \tag{50}$$

Notice that the point $p = \frac{1}{2}$, where an extra symmetry appears is therefore a true critical point. The physics of the model changes drastically as we go through this point:

For $p > \frac{1}{2}$, $\alpha\phi \rightarrow \infty$, means $V^{(p)}_{eff} \rightarrow \infty$, while if $p < \frac{1}{2}$, $\alpha\phi \rightarrow \infty$, means $V^{(p)}_{eff} \rightarrow 0$

This type of "square root gauge theory" has been investigated for the case of a vector potential gauge field theory [16,17], i.e. for an L of the form $\sqrt{F^{\mu\nu}F_{\mu\nu}}$, which has identical transformation properties under conformal transformations of the metric as $y^{\frac{1}{2}}$, with y defined as in (26) and where

no transformation of the gauge fields themselves is assumed. Also these theories have been studied from the point of view of higher dimensional physics in the modified measure formalism [18].

7 Strings and Branes

In the case of strings, we can replace in the Polyakov action, the measure $\sqrt{-\gamma}d^2x$ (where γ_{ab} is the metric defined on the world sheet, $\gamma = det(\gamma_{ab})$ and a, b indices for the world sheet coordinates) by Φd^2x, where $\Phi = \varepsilon^{ab}\varepsilon_{ij}\partial_a\varphi_i\partial_b\varphi_j$. Then for the bosonic string [19], we consider the action

$$S = -\int d\tau d\sigma \Phi[\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{\varepsilon^{ab}}{\sqrt{-\gamma}}F_{ab}] \tag{51}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ and A_a is a gauge field defined in the world sheet of the string. The term with the gauge fields is irrelevant if the ordinary measure of integration is used, since in that case it would be a divergence, but is needed for a consistent dynamics in the modified measure reformulation of string theory. This is due to the fact that if we avoid such a contribution to the action, one can see that the variation of the action with respect to γ^{ab} leads to the vanishing of the induced metric on the string. The equation of motion obtained from the variation of the gauge field A_a is $\varepsilon^{ab}\partial_a(\frac{\Phi}{\sqrt{-\gamma}}) = 0$. From which we obtain that $\Phi = c\sqrt{-\gamma}$ where c is a constant which can be seen is the string tension. The string tension appears then as an integration constant and does not have to be introduced from the beginning. The string theory Lagrangian in the modified measure formalism does not have any fundamental scale associated with it. The gauge field strength F_{ab} can be solved from a fundamental constraint of the theory, which is obtained from the variation of the action with respect to the measure fields φ_j and which requires that $L = M = constant$. Consistency demands $M = 0$ and finally all the equations are the same as those of standard bosonic string theory.

Extensions to both the super symmetric case [20] and to higher branes are possible [19,20]. When considering the modified measure reformulation of the super string, it is very useful to consider the Siegel reformulation of the Green Schwarz action, where the Wess-Zumino term is the square of super symmetric currents [21]. Then the modified measure action will be given by

$S = \int d^2x \Phi L$, where L is given by

$$L = \frac{1}{2}\gamma^{ab} J^{\mu}_{a} J_{\mu b} - i\frac{\varepsilon^{ab}}{\sqrt{-\gamma}} J^{\alpha}_{a} J_{\alpha b} \tag{52}$$

Here the super symmetric currents are defined by $J^{\alpha}_{a} = \partial_a \theta^{\alpha}$, $J^{\mu}_{a} = \partial_a X^{\mu} - i\partial_a \theta^{\alpha} \Gamma^{\mu}_{\alpha\beta} \theta^{\beta}$ and finally the current $J_{\alpha a}$, that contains all the dependence on the compensating fields ϕ_{α} introduced by Siegel [21] to achieve super symmetry invariance (and not just super symmetry up to a total divergence of the usual formulation), is defined as

$$J_{\alpha a} = \partial_a \phi_{\alpha} - 2i(\partial_a X^{\mu})\Gamma_{\mu\alpha\beta}\theta^{\beta} - \frac{2}{3}(\partial_a \theta^{\beta})\Gamma^{\mu}_{\beta\delta}\theta^{\delta}\Gamma_{\mu\alpha\epsilon}\theta^{\epsilon} \tag{53}$$

Then, as opposed to the Siegel case, due to the use of the modified measure, the compensating fields ϕ_{α}, do not enter in the action as in a total divergence (that is they are dynamically irrelevant). Instead, these compensating fields are responsible for the existence of the gauge field A_a, which we explained for the case of the bosonic string and which does not have to be introduced independently of the Siegel compensating fields and in fact it can be read of from the above action to be $A_a = i\theta^{\alpha}\partial_a \phi_{\alpha}$. The gauge field needed for consistent dynamics is now a composite structure of a nature reminiscent to those considered in Ref.22.

For the case of higher p-branes, in the bosonic case, a term of the form $\frac{\varepsilon^{a_1 a_2 \ldots a_{p+1}}}{\sqrt{-\gamma}}\partial_{[a_1} A_{a_2 \ldots a_{p+1}]}$ has to be considered instead of the $\frac{\varepsilon^{ab}}{\sqrt{-\gamma}} F_{ab}$ considered for the string. As in the case of strings, the variation with respect to the φ_j fields requires $L = M = constant$. For higher branes however, a consistent dynamics is achieved as long as $M \neq 0$. Once this is done, in contrast to the standard formulation of Polyakov-type actions for branes, no explicit cosmological term in the brane has to be added. It is interesting to recall at this point that the original motivation [1] for the use of a modified measure was in the context of the cosmological constant problem in field theoretical approaches to gravitation and in the context of the theory of extended objects the modified measure approach continues to be useful concerning how to handle (which in this case means avoiding) the cosmological constant defined in the world brane. Super branes can also be formulated with the use of a modified measure. In this case the Bergshoeff-Sezgin formalism [23], which generalizes for the case of higher branes the Siegel formalism [21] has

to be used. If we do this, we find again that the field $A_{a_1 \ldots a_p}$, which is required for a consistent dynamics in the modified measure formalism, does not have to be introduced separately, instead it appears as a consequence of the Bergshoeff-Sezgin formalism [20]. Again, the Bergshoeff-Sezgin compensating fields are dynamically relevant, unlike the situation in the standard case [23]. Furthermore, in contrast to the treatment of Ref.23, no explicit cosmological term in the brane needs to be included. As it was explained in the string case, the brane tension appears as an integration constant of the equations of motion. Once again, no fundamental scales need to be introduced in the original action of the theory.

8 Discussion and Conclusions

Here we have seen that the consideration of a volume element independent of the metric allows i) to handle the cosmological constant problem, ii) to produce new realistic gravitational theories which allow spontaneous symmetry breaking (ssb) of scale invariance iii) that this ssb of scale invariance does not necessarily requires the existence of Goldstone bosons and iv) that string and brane theories without a fundamental scale are possible.

Concerning the gravitational theory, one should notice that it was crucial to get the physical content of the theory to go to the metric $\overline{g}_{\mu\nu} = \chi g_{\mu\nu}$, where the theory takes the Einstein form.

It has been the subject of great debate which conformal frame is the physical one. For a review, see Ref.24.

In our case it appears that $\overline{g}_{\mu\nu} = \chi g_{\mu\nu}$ is indeed the correct choice.

This is more transparent if we look at the theory in the Hamiltonian formalism. In Hamiltonian languge the quantization of the theory and the statistical mechanichs phase space are more directly available.

In this case it is immediate to see that the original metric $g_{\mu\nu}$ has vanishing canonically conjugate momenta. In contrast to this, it is a simple matter to see that all the canonically conjugate momenta to the connections $\Gamma^{\lambda}_{\mu\nu}$ are functions of $\overline{g}_{\mu\nu} = \chi g_{\mu\nu}$. So that $\Gamma^{\lambda}_{\mu\nu}$ and $\overline{g}_{\mu\nu}$ are conjugate variables, while the original metric does not have a conjugate variable.

Furthermore, very much connected to this, in the quantum theory, it is the expansion of $\overline{g}_{\mu\nu}$ in creation and annihilation operators (after suitable gauge fixing) the one that provides us with the correctly normalized graviton states.

In contrast the original metric has unphysical singularities for $V + M = 0$ in the two measure model for example, while no such a problem appears in the Einstein frame. This shouldn't worry us, in phase space we don't even see the original metric.

Another issue for further study is the connection of the modified measure formalism to of brane scenarios. It appears for example that the two measure formalism may be related to such type of scenario, which automatically assigns a different measure for the bulk and for the brane [25].

Furthermore, the quintessence picture (section 6) presents some remarkable points of contact with what is subsequently discussed in connection to the modified measure formulation of brane theories in that both require the use of antisymmetric potentialsfor achieving consistent dynamics. This point also deserves further research.

Another direction which seems to have a more direct connection to what is done here is the discrete Kaluza Klein approach developed in Ref.26 , where two copies of space-time are introduced in order to develop a non commutative geometry. In a contex like this the measure Φ can naturally arise as the jacobian of the mapping of these two spaces.

Finally let us point out that many of the results obtained here agree with the considerations of Bekenstein and Meisels [27], which were considering local rather than global scale invariance. In any case, they found that local scale invariance goes together with a constant vacuum energy and constant particle masses, exactly what it is found here in the Einstein frame in the unbroken sector of the theory. For example when M can be ignored in the two measure theory. In the quintessence scale invariant model the analysis is more complex because there are now two sources of scale symmetry breaking (M and ω) . This will be studied further in the future [15].

9 Acknowledgements

I would like to thank J.Alfaro, M. Banados, J.Bekenstein, N.Berkovits, Z.Bern, L. Cabral, S. del Campo, C.Castro, F. Cooperstock, A.Davidson, M.Dine, P.Gaete,M.Giovannini, F.Hehl, L.Horwitz, A.Kaganovich, M.Loewe, Y.Ne'eman, M.Pavsic, J.Portnoy, E.Spallucci, J.Schwarz, Y.Verbin and K.Wali for conversations on the subjects of this paper.

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